Painlevé Integrability and Abundant Localized Structures of (2+1)-dimensional Higher Order Broer-Kaup System

Ji Lin[a,b] and Hua-mei Li[b]

[a] Physics Department, Shanghai Jiao Tong University, Shanghai, 200030, China
[b] Department of Physics, Zhejiang Normal University, Jinhua 321004, China

Reprint requests to J. L.; E-mail:jhlinji@mail.jhptt.zj.cn

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It is proven that the (2+1) dimensional higher-order Broer-Kaup system the possesses the Painlevé property, using the Weiss-Tabor-Carnevale method and Kruskal's simplification. Abundant localized coherent structures are obtained by using the standard truncated Painlevé expansion and the variable separation method. Fractal dromion solutions and multi-peakon structures are discussed. The interactions of three peakons are investigated. The interactions among the peakons are not elastic; they interchange their shapes but there is no phase shift.

Key words: Painlevé Analysis; Variable Separation Method; Fractal Dromion Solution; Peakon Structure.

1. Introduction

The integrability of a nonlinear partial differential equation (NPDE) is an interesting topic in nonlinear science. Many methods have been established by mathematicians and physicists to study the integrability of NPDE's. Some of the most important methods are the inverse scattering method(IST) [1], the bilinear method [2], the symmetry reductions [3], Bäcklund and Darboux transformations [4], the Painlevé analysis method [5], etc. As known, a model which is integrable in one sense may not be integrable in another sense. For instance, some Lax integrable models may not be Painlevé integrable [6]. Therefore, when saying a model is integrable, we must say under what specific meaning(s). For example, we say a model is Painlevé integrable if the model has the Painlevé property, and a model is Lax or IST integrable if the model has a Lax pair and can be solved by the IST approach.

To find some exact, physically significant coherent soliton solutions (which are localized in all directions) in 2+1 dimensions is much more difficult than in 1+1 dimensions, though there are many useful methods developed to find solutions of 1+1 dimensional PDE's. Recently, one special type of exponentially localized solitons called dromions driven by some straight ghost line solitons or by some straight ghost line solitons together with some curved line solitons for many 2+1 dimensional integrable models has been found, using different approaches [7 - 10]. From the study of the symmetry we know that there exist quite rich symmetry structures for the 2+1 dimensional integrable models [11]. This implies that 2+1 dimensional integrable models may have quite abundant structures. Recently we have found more and more localized coherent structures (such as ring solitons, breathers, solitoffs, lumps, instantons and fractals) of 2+1 dimensional integrable models such as DS (Davey-Stewartson), NNV (Nizhnik-Novikov-Veselov), ANNV (asymmetric NNV), (2+1)-dimensional AKNS equations using the variable separation method [12].

In this paper, we combine the standard Weiss-Tabor-Carnevale (WTC) approach and Kruskal's simplification to study the Painlevé integrability of the 2+1 dimensional higher order Broer-Kaup (HBK) system in Section 2. In Sect. 3, we investigate exact localized coherent structures of the HBK system using the variable separation method. We report on fractal structures and multi-peakon solutions of the HBK system and interactions of peakons in Section 4. The last section gives a short summary and discussion.

2. Painlevé Property of 2+1-dimensional HBK System

The 2+1 dimensional HBK system

$$u_{yt}+4(u_{xx}+u^3-3uu_x+3uw)_{xy}+12(uv)_{xx}=0, \quad (1)$$

$$v_t+4(v_{xx}+3vu^2+3uv_x+3vw)_x=0, \quad (2)$$

$$w_y-v_x=0 \quad (3)$$

was first obtained from the inner parameter dependent symmetry constraints of the KP equation [13]. When we take $y=x$, the system (1) - (3) is reduced to the usual (1+1)-dimensional HBK system.

One of the most powerful methods to prove the integrability of a model is the so-called Painlevé analysis developed by WTC [5]. Furthermore, the Painlevé analysis can also be applied to find out some exact solutions, no matter whether the model is integrable or not. If one needs only to prove the Painlevé property of a model, one may use Kruskal's simplification for the WTC approach [14]. If one hopes to find some more information about the model, one has to use the original WTC approach or some of its extended forms [10, 15, 16].

According to the standard WTC method, if the 2+1 dimensional HBK system is Painlevé integrable, then all the possible solutions of the system can be represented as

$$u=\sum_{k=0}^{\infty}u_kf^{k+\alpha},\ v=\sum_{k=0}^{\infty}v_kf^{k+\beta}, \quad (4)$$

$$w=\sum_{k=0}^{\infty}w_kf^{k+\gamma},$$

with sufficiently many arbitrary functions u_k, v_k, w_k in addition to f, where $f=f(x,y,t)$, $u_k=u_k(x,y,t)$, $v_k=v_k(x,y,t)$, and $w_k=w_k(x,y,t)$ are analytical functions in the neighborhood of $f=0$, and α, β and γ should all be the negative integers. In other words, the solutions of the system are single valued on an arbitrary singularity manifold.

Inserting (4) in (1) - (3), a leading-order analysis uniquely gives

$$\alpha=\gamma=-2,\ \beta=-1, \quad (5)$$

$$u_0=-f_xf_y,\ v_0=f_x,\ w_0=-f_x^2. \quad (6)$$

Consequently, putting to zero all coefficients of the powers f^k, we can obtain the recursion relations to determine the functions u_k, v_k, w_k:

$$4(k-3)(k-4)f_x^2((k-5)(k-1)f_xf_yu_k+3f_xv_k+3f_yw_k)$$
$$=U_k(u_i,v_i,w_i,...,i\le k-1), \quad (7)$$

$$4k(k-2)(k-4)f_x^3v_k-12(j-4)f_x^2f_yw_k$$
$$=V_k(u_i,v_i,w_i,...,i\le k-1), \quad (8)$$

$$-(k-2)f_xv_k+(k-2)f_yw_k$$
$$=W_k(u_i,v_i,w_i,...,i\le k-1), \quad (9)$$

where $U_k(u_i,v_i,w_i,...)$, $V_k(u_i,v_i,w_i,...,)$ and W_k $(u_i,\ v_i,\ w_i,...)$ are some complicated functions of $f_x,f_y,...,u_0,u_1,...,v_0,v_1,...,w_0,w_1,...,w_{k-1}$.

From (7) - (9), putting to zero the coefficient determinant of $\{u_k,v_k,w_k\}$, we find that the resonances occur at

$$k=-1,\ 1,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5. \quad (10)$$

The resonance at $k=-1$ corresponds to that the singularity manifold f is an arbitrary function. If the HBK system is Painlevé integrable, the resonance conditions at $k=1,2,3,3,4,4,5$ must be satisfied identically such that the seven arbitrary functions among the u_k,v_k,w_k can be introduced into the general series expansions (4). For $k=1$, we can easily obtain from (7) - (9)

$$v_1=f_{xy},\ w_1=f_{xx}, \quad (11)$$

while u_1 is an arbitrary function. For $k=2$, (7) - (9) give us

$$u_2=\frac{1}{f_y}(-u_{1y}+v_2), \quad (12)$$

$$w_2=-\frac{1}{12f_x}(f_t+4f_{xxx}+12u_1^2f_x+12u_1f_{xx}),$$

and v_2 is arbitrary. For $k=3$, from (7) - (9), we get

$$v_3=-\frac{1}{12f_x^3}(f_{ty}f_x+4f_{xxxy}f_x+12f_xu_1f_{xxy} \quad (13)$$
$$+24u_1f_x^2u_{1y}-4f_{xy}f_{xxx}+12f_x^2v_{2x}-f_tf_{xy}$$
$$+12u_{1y}f_{xx}f_x-12u_1f_{xx}f_{xy}-12f_x^2f_yw_3),$$

and $u_3,\ w_3$ are arbitrary. For $k = 4$, we have

$$w_4 = \frac{1}{2f_y}(v_{3x} - w_{3y} + 2v_4 f_x), \quad (14)$$

while $u_4,\ v_4$ also are arbitrary. For $k = 5$, because the resonance conditions are complicated, we use Kruskal's simplification, *i. e.*:

$$f = x + \psi(y,t),\ u_k = u_k(y,t), \quad (15)$$
$$v_k = v_k(y,t),\ w_k = w_k(y,t).$$

where $\psi(y,t)$ is an arbitrary function of $\{y,t\}$ only. One can use this simplification without loss of generality to verify the Painlevé integrability. Under the simplification (15), the above results can be simplified as

$$u_0 = -\psi_y,\ v_0 = -w_0 = 1,\ v_1 = w_1 = 0, \quad (16)$$

$$u_2 = \frac{1}{\psi_y}(-u_{1y} + v_2),\ w_2 = -\frac{1}{12}(\psi_t + 12u_1^2), \quad (17)$$

$$v_3 = -\frac{1}{12}(\psi_{yt} + 24u_1u_{1y} - 12w_3\psi_y), \quad (18)$$
$$w_4 = \frac{1}{2\psi_y}(2v_4 - w_{3y}).$$

Using the simplification (15) and the relations (16) - (18) and with the help of the computer algebras (say Maple) we can get

$$\begin{aligned} v_5 = {} & \frac{1}{48}(-48u_1v_4 - 24v_2u_1u_2 - 24u_3v_2 - 36u_2v_3 \\ & - 12v_2w_3 - 12v_3w_2 - v_3\psi_t - 12v_3u_1^2 - v_{2t}) \\ & + \frac{1}{48\psi_y^2}(24u_2u_3\psi_y^3 + 24u_1u_4\psi_y^3 - 2\psi_{yy}w_{3y} \\ & + 2\psi_y w_{3yy} - 4\psi_y v_{4y} + 4\psi_{yy}v_4), \end{aligned} \quad (19)$$

$$\begin{aligned} w_5 = {} & \frac{1}{12\psi_y}(48u_1v_4 + 24v_2u_1u_2 + 24u_3v_2 \\ & + 36u_2v_3 + 12v_2w_3 - 24u_2u_3\psi_y - 24u_1u_4\psi_y \\ & + 12v_3w_2 + v_3\psi_t + 12v_3u_1^2 + 60v_5 + v_{2t}), \end{aligned} \quad (20)$$

and u_5 is an arbitrary function from the resonance conditions for $k = 5$. All of the resonance conditions with eight arbitrary functions are satisfied identically, so that the 2+1 dimensional HBK system (1 - 3) is Painlevé integrable.

3. Variable Separation Method

According to the standard truncated Painlevé expansion we have the Bäcklund transformation

$$u = \frac{f_x}{f} + u_0,\ v = -\frac{f_xf_y}{f^2} + \frac{f_{xy}}{f} + v_0, \quad (21)$$
$$w = -\frac{f_x^2}{f^2} + \frac{f_{xx}}{f} + w_0,$$

where $u_0,\ v_0,\ w_0$ are the seed solution of the (2+1) dimensional HBK system. In order to better understand the localized coherent structures of the 2+1 dimensional HBK system, we find it useful to apply the variable separation method to this system. According to the idea of the variable separation method [12], we assume that

$$f = 1 + a_1h(x,t) + a_2q(y,t) + Ah(x,t)q(y,t), \quad (22)$$

where $a_1,\ a_2$ and A are arbitrary constants and $h \equiv h(x,t),\ q \equiv q(y,t)$ are only functions of $\{x,t\},\{y,t\}$, respectively, and take the seed solution as

$$u_0 \equiv u_0(x,t),\ v_0 = 0,\ w_0 \equiv w_0(x,t). \quad (23)$$

It is clear that the variables x and y have now been separated totally into the functions h and q, respectively. Substituting the Bäcklund transformation (21) and the ansatz (22) with (23) into (1) - (3) and using computer algebras (Maple or Mathematica), we obtain

$$\begin{aligned} & (2(a_1 + Aq) - fh_x^{-1}\partial_x)(h_t + 4h_{xxx} + 12u_0h_{xx} \\ & + 12(w_0 + u_0^2)h_x) + (2(a_2 + Ah) - fq_y^{-1}\partial_y)q_t = 0. \end{aligned} \quad (24)$$

Because h is y-independent and q is x-independent, we can separate (24) into the following two equations

$$\begin{aligned} & h_t + 4h_{xxx} + 12u_0h_{xx} + 12(w_0 + u_0^2)h_x - c_0 - c_1h \\ & - (a_1c_1 + Ac_3 - a_1a_2c_3 - a_1^2c_0)h^2 = 0, \end{aligned} \quad (25)$$

$$\begin{aligned} q_t = {} & c_3 + (-c_1 + 2a_2c_3 + 2a_1c_0)q \\ & + (-a_2c_1 + a_2^2c_3 + a_2a_1c_0 + Ac_0)q^2, \end{aligned} \quad (26)$$

where the functions $c_0 \equiv c_0(t)$, $c_1 \equiv c_1(t)$ and $c_3 \equiv c_3(t)$ are introduced by the variable separation approach.

To find the general solution of (25) for any fixed u_0, w_0 is still quite difficult. Fortunately we know that the seed solutions u_0, w_0 are arbitrary functions of x and t, so we can treat this problem alternatively by considering h to be an arbitrary function of x and t and determining the function w_0 by (25). The result is

$$w_0 = -u_0^2 - \frac{1}{12h_x}(h_t + 4h_{xxx} + 12u_0h_{xx} - c_0 - c_1h - (a_1c_1 + Ac_3 - a_1a_2c_3 - a_1^2c_0)h^2). \tag{27}$$

To solve the Riccati equation (26) is very easy because of the arbitrariness of the functions c_0, c_1 and c_3. We can get the solution of (26)

$$q = \frac{g_1}{g_3 + p} + g_2, \tag{28}$$

where $g_1 \equiv g_1(t)$, $g_2 \equiv g_2(t)$, $g_3 \equiv g_3(t)$ and $p \equiv p(y)$ can all be considered as arbitrary functions of the indicated variables, while c_0, c_1 and c_3 are related to $g_1 \equiv g_1(t)$, $g_2 \equiv g_2(t)$ and $g_3 \equiv g_3(t)$ by

$$c_0 = \Big[- a_2^2g_1g_{2t} - (-2a_2g_2 - a_2^2g_2^2 - 1)g_{3t} - (-a_2^2g_2 - a_2)g_{1t}\Big] \Big/ (a_2a_1 - A)g_1, \tag{29}$$

$$c_1 = \Big[- 2a_2Ag_1g_{2t} - (-2a_2a_1g_2 - 2Aa_2g_2^2 - 2Ag_2 - 2a_1)g_{3t} - (-a_2a_1 - 2Aa_2g_2 - A)g_{1t}\Big] \Big/ (a_2a_1 - A)g_1, \tag{30}$$

$$c_3 = \frac{g_1g_{2t} - g_2^2g_{3t} - g_2g_{1t}}{g_1}. \tag{31}$$

Now we insert (21) with (23) into the 2+1 dimensional HBK system and get the exact solutions of the HBK system

$$u = \frac{(a_1 + Aq)h_x}{1 + a_1h + a_2q + Ahq} + u_0, \tag{32}$$

$$v = \frac{(A - a_2a_1)h_xq_y}{(1 + a_1h + a_2q + Ahq)^2}, \tag{33}$$

$$w = \frac{(a_1 + Aq)h_{xx}}{1 + a_1h + a_2q + Ahq} - \frac{(a_1 + Aq)^2h_x^2}{(1 + a_1h + a_2q + Ahq)^2} + w_0, \tag{34}$$

where h and u_0 are arbitrary functions of $\{x, t\}$, q is given by (28) and w_0 is determined by (27) with (29) - (31).

4. Special Localized Structures of a 2+1-dimensional HBK System

Because (32) - (34) contain the arbitrary function h, there are abundant soliton structures following from them, such as the multi-solitoff solutions, the multi-dromion structure driven by multiple straight lines and curved line ghost solitons, single- (or multi-) curved line (camber) solitons, multi-lump structure, breathers, instantons, etc., given in [12, 18]. Because the field v expansion (33) is the same as that of the counterparts for the DS, NNV, and AKNS equations, we do not discuss here these types of localized coherent structures further. In this paper, we focus on fractal structures and peakon solutions.

4.1. Fractal Structure

(i) Fractal dromions

In 2+1 dimensions, one of the most important localized coherent structures ,which is the so-called dromion solution that is localized exponentially in all directions, is investigated. Recently, some types of piecewise smooth solutions like the peakons, cuspons and compactons are widely applied in 1+1 dimensional soliton systems [19]. All these lower dimensional piecewise smooth functions can also be used to construct higher dimensional peakons, cuspons and compactons. If we select h in (33) as some types of lower dimensional piecewise smooth functions with fractal structures, we can construct exact solutions of the 2+1 dimensional HBK system. For instance, if we select ($\xi = x - t$)

$$h = \exp\big\{|\xi|(3/2 + \sin(\ln \xi^2))\big\}, \tag{35}$$
$$q = A(t)\exp|y\big\{|(3/2 + \sin(\ln y^2))\big\}$$

in (32) - (34), we obtain an exact solution of the 2+1 dimensional HBK system with fractal structure

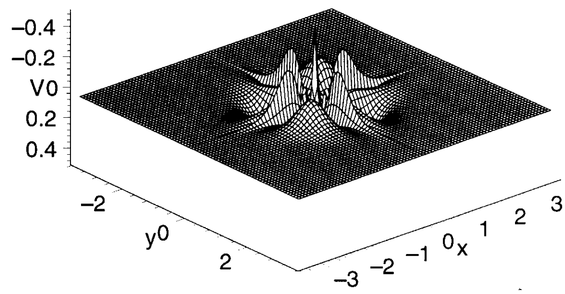

Fig. 1. Plot of the fractal dromion solution of the HBK system for the field v (33) with (35) and (36).

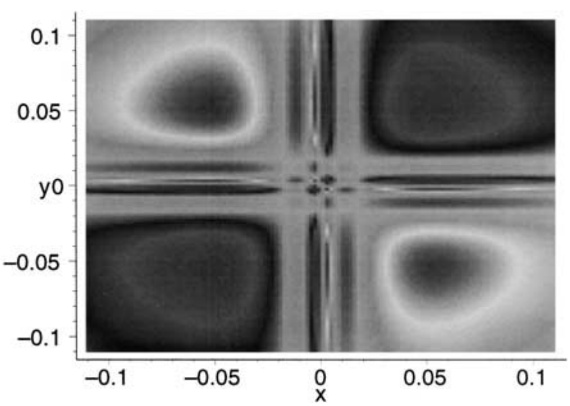

Fig. 2. The density of the fractal dromion (in Fig. 1) in the intervals $\{x \in [-0.11,\ 0.11],\ y \in [-0.11,\ 0.11]\}$.

for small $x - t$ and y. Figure 1 shows the dromion structure of the field v in (33). Figure 2 shows the density plot of the dromion structure for the field v in the region $\{x \in [-0.11, 0.11], y \in [-0.11, 0.11]\}$ for $t = 0$. If we plot the structure of the solution in smaller regions, say, $\{x \in [-0.0044, 0.0044], y \in [-0.0044, 0.0044]\}$ or $\{x \in [-0.000008, 0.000008], y \in [-0.000008, 0.000008]\}$,..., $\{x \in [-6.4\times10^{-10}, 6.4\times10^{-10}], y \in [-6.4\times10^{-10}, 6.4\times10^{-10}]\}$,..., we can get the interesting result that there are completely similar structures, as shown in Figure 2.

(ii) Fractal multi-dromions

In addition to the regular fractal functions, there exist some other types of irregular fractal functions. One of the famous fractal functions is the Weierstrass function

$$h_1(\xi) = \sum_{k=0}^{N} (\frac{3}{2})^{-k/2} \sin((\frac{3}{2})^k(\xi)),\ N \to \infty. \quad (36)$$

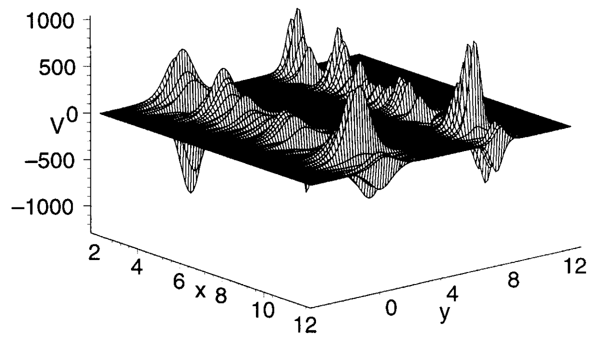

Fig. 3. Plot of the fractal multi-dromion solution of the HBK system for the field v (33) with (37) and (38) for $t = 0$.

If we select h and q as

$$h = 2 + \frac{1}{2}h_1 \tanh(4x + t - 20), \quad (37)$$

$$q = \frac{1}{5}\tanh(y) + \frac{1}{4}\tanh(2y - 15), \quad (38)$$

we can obtain a typical fractal multi-dromion structure of the field v and show this structure for $t = 0$ in Figure 3. No matter how small or large the ranges are, one can find the amplitudes of the multi-dromion always to change stochastically.

4.2. Peakon Structures

Since the peakon solution ($u = c\exp(-|x - ct|)$), which is called a weak solution of the (1+1) dimensional Camassa-Holm equation [19] due to its discontinuity at the crest, was given already, the properties and interaction behaviors of the peakons for 1+1 dimensional integrable models have been attracted much attention of physicists and mathematicians [20]. The collisions among the 1+1 dimensional peakons are completely elastic [21]. However, to our knowledge, the higher dimensional peakon solutions are not yet investigated, because the 2+1 dimensional two peakon solutions for the 2+1 AKNS were obtained only recently [22]. The collision of two 2+1 dimensional peakons is inelastic. For the 2+1 dimensional HBK system, ff we select

$$h = 1 + \sum_{i=1}^{M} b_i \begin{cases} \exp(k_i x - \omega_i t + x_{0i}), \\ \qquad k_i x - \omega_i t + x_{0i} \le 0, \\ -\exp(-k_i x + \omega_i t - x_{0i}) + 2, \\ \qquad k_i x - \omega_i t + x_{0i} > 0, \end{cases} \quad (39)$$

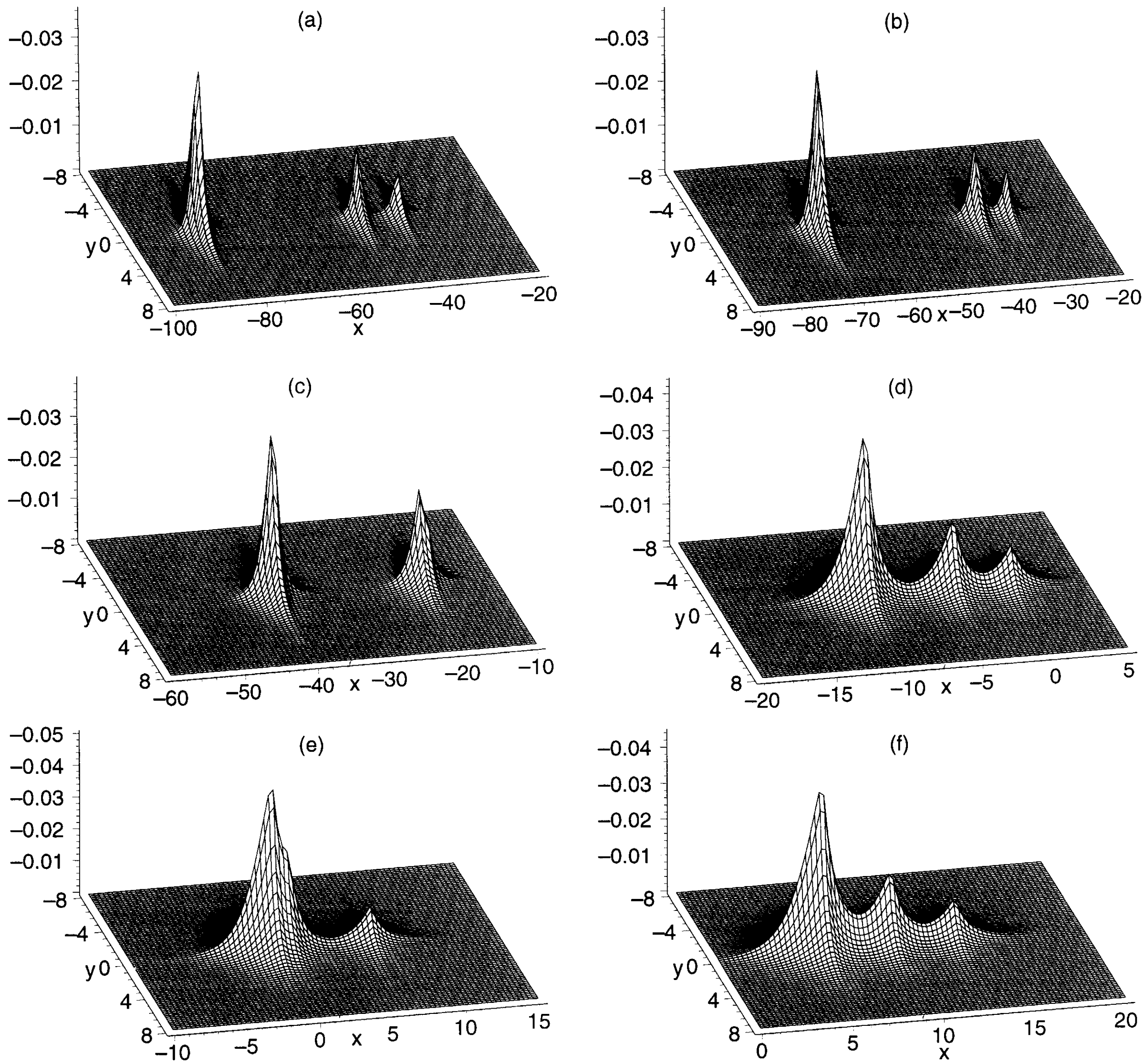

Fig. 4a - f. Plot of the interaction of three peakons for the field v given by (33) with (39), (40) and (41) at times (a) $t = -50$, (b) $t = -40$, (c) $t = -20$, (d) $t = 0$, (e) $t = 6.5$, (f) $t = 13$.

and the function q as a special type of a static weak solution

$$q = \sum_{i=1}^{N} c_i \begin{cases} \exp(l_i y), & l_i y \le 0, \\ -\exp(-l_i y) + 2, & l_i y > 0, \end{cases} \tag{40}$$

respectively, where b_i, c_i, k_i, ω_i, l_i and x_{0i} are all arbitrary constants and M and N are arbitrary integers, we can obtain multi-peakon structures of the field v in (33). For instance, we take the coefficients in (39) and (40) as

$$M = 3,\ N = 1,\ k_1 = k_2 = k_3 = b_1 = b_2 = b_3 = 1, \tag{41}$$
$$c_1 = a_1 = a_2 = \omega_1 = 1, A = \frac{1}{9},\ 2\omega_2 = 4\omega_3 = 3,$$
$$x_{01} = 0,\ x_{02} = 10,\ x_{03} = 4,$$

and obtain the three peakons structure solution of the field v. Figure 4. shows the interaction behavior

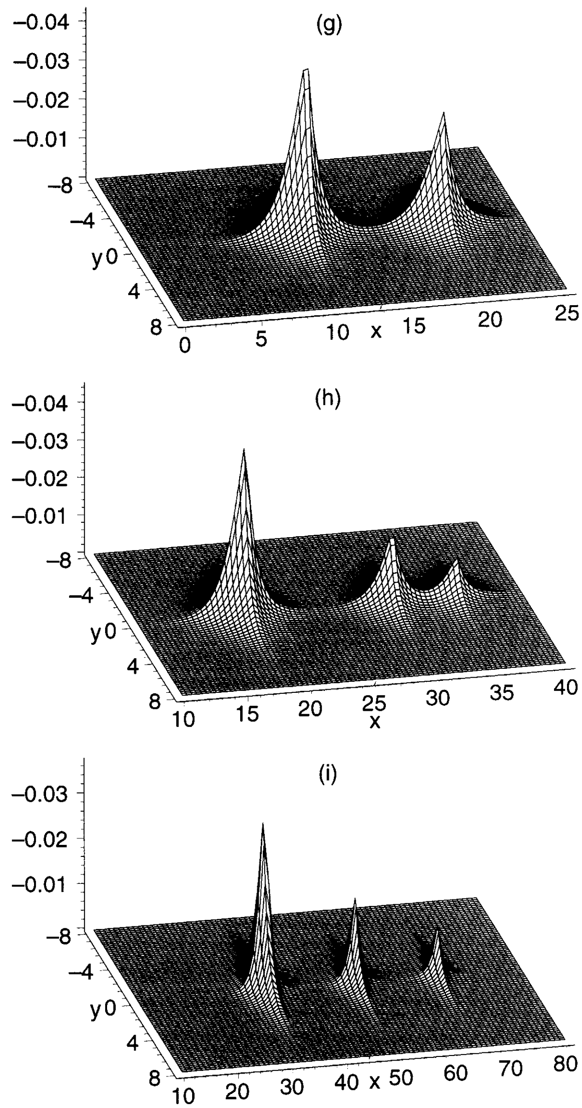

Fig. 4g - i. (g) t = 20, (h) t = 30, (i) t = 50.

of the three peakons. From Fig. 4(a) to (i) we observe three collisions among the three peakons in the course of time. The middle peakon passes through the small one and goes forward after the first collision, but they exchange their shapes, i. e. the shape of the middle peakon becomes that of the small one and *vice versa*. The second collision occurs between the large peakon and the small one (which is the original middle peakon).

The property of the second interaction is the same as that of the first one, and so the third is. After three interactions, the large peakon has become the small one while the small peakon has become the large one, but the middle peakon returns to its original shape through the two collisions. We can also show that the velocity of the motion of any peakon does not change because of the interaction with other peakons. Figure 5 is a plot of the evolution of the centers (at y = 0) of the three peakons which are the same as in Figure 4. We notice that there are not any phase shifts for interacting peakons.

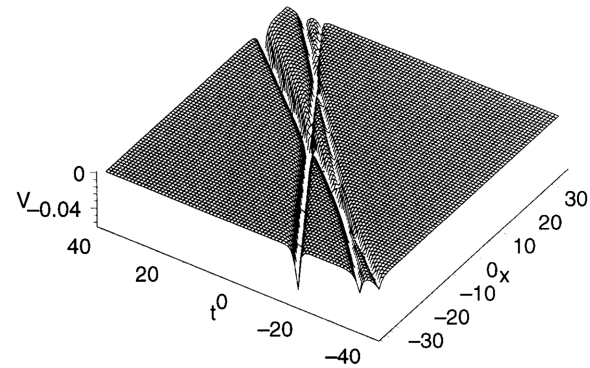

Fig. 5. Plot of the evolution of the interacting peakon centers (located at y = 0) of v with (39) and (40).

5. Summary and Discussions

In summary, applying the standard WTC approach and Kruskal's simplification to the 2+1 dimensional HBK system, we showed that this system satisfies the Painlevé property. Starting from a Bäcklund transformation and using the variable separation method, we obtain generalized solutions (which include the arbitrary functions appearing in the seed solution and in the variable separation procedure) of the 2+1 dimensional HBK system. Many localized coherent structures can be detected, such as multi-solitoff solutions, multi-dromion structures, ring soliton solutions, breathers, instantons, which are reported in [12, 17], etc. In addition to many types of stable localized structures, one obtaines many types of fractal patterns by selecting the arbitrary functions as fractal solutions of some lower dimensional nonintegrable models. We describe some types of exact fractal solutions such as the fractal dromions and the fractal multi-dromions. Because of the wide applications of the fractal theory and the soliton theory in mathematics, physics, biology etc., studying the more fractal solutions of the integrable equations with nonintegrable boundary and / or initial conditions such as the stochastic and fractal boundary and / or initial conditions will interest physicists and mathematicians. Actually, the 2+1 dimensional HBK system is both Lax integrable and Painlevé integrable. Whereas, as we know, the stochastic and fractal properties are related to nonintegrable phenomena, one can find that the initial and / or boundary conditions possess also fractal properties if one considers the boundary and / or initial conditions of the fractal solutions obtained here.

In this paper we have also reported on abundant multi-peakon structures and have investigated the interactions (or collisions) of three peakons. It is known that the travelling dromions and the travelling saddle type ring soliton solutions possess completely elastic interaction properties [12], while the interactions among the travelling peakons given here and in [22] are not elastic. After the interaction the peakons exchange their shapes completely, while their travelling speeds do not change and there are no phase shifts. Though the 1+1 dimensional peakon solutions have been investigated extensively in the literature, the 2+1 dimensional peakon structures are less known. So the higher dimensional peakon solutions of the various models are worth to be studied further.

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